

The “mean king’s problem” with continuous variables

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We present the solution to the “mean king’s problem” in the continuous variable setting. We show that in this setting, the outcome of a randomly-selected projective measurement of any linear combination of the canonical variables (\hat{x}, \hat{p}) can be ascertained with arbitrary precision. Moreover, we show that the solution is in turn a solution to an associated “conjunctive” version of the problem, unique to continuous variables, where the inference task is to ascertain all the joint outcomes of a simultaneous measurement of any number of linear combinations of (\hat{x}, \hat{p}) .

A hallmark of the emergence of quantum information has been the shift in foundational research towards probing interpretational aspects of quantum mechanics through information-theoretic tasks, the feasibility of which is to be decided within standard quantum theory. The existence or non-existence of solutions to these tasks has proved invaluable not only in revealing unsuspected implications of quantum mechanics, but also in refining our general heuristics of the theory, a good part of which inherits from the early complementarity arguments of the “old” quantum theory.

One such task is what has come to be known as the “mean king’s problem”. In its original version [1] (and as retold in the whimsical setting of Refs. [2, 3]), a physicist, Alice, is challenged by a mean king to precisely ascertain the outcome of an ideal measurement that the king performs of a spin-1/2 observable randomly chosen from the mutually complementary set $\{\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$. The conditions are that Alice can access the system both before and after the king’s measurement, and that only at the end will the king reveal the observable that was actually measured and summon her to ascertain the corresponding measurement outcome. While such a task stands in overt defiance of the complementarity heuristics, it was nevertheless shown in [1] that the task is indeed feasible according to quantum theory, with the solution involving initial and final measurements by Alice performed on a composite system of the spin and some other quantum particle. The extension of the problem to arbitrary-dimensional discrete Hilbert spaces has also been solved; first, for prime dimensions [4], and later, through a connection to mutually-unbiased bases (MUB) [5], for prime-power dimensions [6]. More recently, the problem was solved for arbitrary discrete dimensions [7] with generalized (POVM) measurements by Alice.

In this letter we present the solution to the mean king’s problem for the continuous variable (infinite-dimensional Hilbert space) case, in which Alice is challenged to infer

the outcome from a randomly chosen element of a continuous set of mutually complementary measurements. The solution we present here is derived from the solution to a closely-related joint measurement inference problem addressed by one of us in a recent publication [8]. Consequently, the mean king’s problem can be thought of as dual to a related inference problem that is unique to the continuous variable setting, and which involves the simultaneous measurements of several complementary operators. We term this dual inference task the “conjunctive” version of the mean king’s problem, and show that the solution of the standard king’s problem is at the same time the solution of its associated conjunctive version.

We begin by stating the continuous variable version of the standard king’s problem. Here, the king challenges Alice to ascertain the outcome of an ideal measurement that he performs of a quadrature observable

$$\hat{X}_\phi = \hat{x} \cos(\phi) + \hat{p} \sin(\phi), \quad (1)$$

where \hat{x} and \hat{p} are canonically-conjugate operators and ϕ is a randomly chosen angle from the interval $[0, \pi)$. More generally, the king can perform any projective measurement that distinguishes the elements of the eigenbasis of \hat{X}_ϕ . Let $|\xi\rangle_\phi$ then be an eigenvector of \hat{X}_ϕ , with eigenvalue ξ , and distinguish the special case of $\hat{X}_0 = \hat{x}$ with the notation $|\xi\rangle_x = |\xi\rangle_0$. The eigenvectors of \hat{X}_ϕ are then given by

$$|\xi\rangle_\phi = \hat{R}(\phi)|\xi\rangle_x, \quad \text{with} \quad \hat{R}(\phi) \equiv e^{-i\frac{\phi}{2}(\hat{p}^2 + \hat{x}^2)}. \quad (2)$$

Alice’s challenge is then to perform initial and final measurements yielding outcomes such that for every element of the continuous family of bases $\{\mathcal{B}(\phi)|\phi \in [0, \pi)\}$, where $\mathcal{B}(\phi) = \{|\xi\rangle_\phi|\xi \in \mathbb{R}\}$, a certain value X_ϕ is guaranteed to have been the result of the king’s measurement. In other words, Alice should be able to assign a probability distribution $P(\xi) = \delta(\xi - X_\phi)$ for some X_ϕ to each $\mathcal{B}(\phi)$ measurement, based on her measurement outcomes.

Note that as in the discrete variable case, the set of possible king's measurements involves mutually complementary observables, or in other words, the basis set $\{\mathcal{B}(\phi)\}$ consists of mutually unbiased bases: for any two vectors from two bases $\mathcal{B}(\phi)$ and $\mathcal{B}(\phi')$, we have

$$\phi' \langle \xi' | \xi \rangle_\phi = {}_x \langle \xi' | \hat{R}(\phi - \phi') | \xi \rangle_x. \quad (3)$$

The matrix element is the Feynman propagator for a harmonic oscillator of unit angular frequency, for a time interval $\phi - \phi'$. Since the modulus of the propagator of any quadratic Hamiltonian is independent of the initial and final coordinates[9], the modulus $|\phi' \langle \xi' | \xi \rangle_\phi|$ is independent of ξ and ξ' .

A solution to the mean king's problem for the basis set $\{\mathcal{B}(\phi)\}$ can then be obtained as follows: Alice will have at her disposal another particle with canonical variables \hat{x}' and \hat{p}' . For the two-particle system, we can define the conjugate pairs $(\hat{x}_\pm, \hat{p}_\pm)$

$$\hat{x}_+ = (\hat{x} + \hat{x}')/2, \quad \hat{p}_+ = \hat{p} + \hat{p}', \quad (4a)$$

$$\hat{x}_- = \hat{x} - \hat{x}', \quad \hat{p}_- = (\hat{p} - \hat{p}')/2. \quad (4b)$$

In particular, she will need to perform initial and final measurements yielding eigenstates of the commuting pair (\hat{x}_+, \hat{p}_-) , which up to normalization are given by

$$|x_+, p_-\rangle = \int_{-\infty}^{\infty} ds e^{isp_-} \left| x_+ + \frac{s}{2} \right\rangle_x \left| x_+ - \frac{s}{2} \right\rangle_{x'}, \quad (5)$$

and eigenstates of the commuting pair (\hat{x}_-, \hat{p}_+) , given by

$$|x_-, p_+\rangle = \int_{-\infty}^{\infty} ds e^{isp_+} \left| s + \frac{x_-}{2} \right\rangle_x \left| s - \frac{x_-}{2} \right\rangle_{x'}. \quad (6)$$

For later convenience, we define the phase factor $e^{i\gamma} \equiv \langle x_-, p_+ | x_+, p_- \rangle = e^{i(x_+ p_+ - x_- p_-)}$. Given the projector $\hat{\Pi}_\phi(\xi) = |\xi\rangle_\phi \langle \xi|$, the probability amplitude $\langle x_-, p_+ | \hat{\Pi}_\phi(\xi) | x_+, p_- \rangle$ can then be written as

$$e^{-i\gamma} \langle x_-, p_+ | \hat{\Pi}_\phi(\xi) | x_+, p_- \rangle = {}_\phi \langle \xi | \hat{\Delta}(x, p) | \xi \rangle_\phi, \quad (7)$$

where we define the labels

$$x \equiv x_+ + x_-/2, \quad p \equiv p_- + p_+/2, \quad (8)$$

and the operator

$$\begin{aligned} \hat{\Delta}(x, p) &= e^{-i\gamma} \text{Tr}'(|x_+, p_-\rangle \langle x_-, p_+|) \\ &= \int_{-\infty}^{\infty} ds e^{isp} \left| x + \frac{s}{2} \right\rangle_x \left\langle x - \frac{s}{2} \right|_x, \end{aligned} \quad (9)$$

with $\text{Tr}'(\dots)$ denoting traces with respect to the primed degree of freedom. The operator $\hat{\Delta}(x, p)$ is the Weyl-symmetrized bivariate δ -function[10],

$$\hat{\Delta}(x, p) = \frac{1}{2\pi} \int_{\mathbb{R}^2} d^2\chi e^{-i(\chi_1(x - \hat{x}) + \chi_2(p - \hat{p}))}, \quad (10)$$

and has a Wigner function that is a δ -function in phase-space. However, being non-positive and non-local in the \hat{x} representation, $\hat{\Delta}(x, p)$ is more accurately interpreted as a generalized parity operator[11] (up to a factor). The connection with the Wigner representation nevertheless proves useful for calculations. In particular, we find for the case $\phi = 0$, the expectation value

$$\langle \xi | \hat{\Delta}(x, p) | \xi \rangle_x = \delta(\xi - x). \quad (11)$$

Similarly, for the general case $\phi \neq 0$, we note that $\hat{R}(\phi)$ in Eq. (2) implements a rotation of the canonical operators, and in turn a rotation of the labels of $\hat{\Delta}(x, p)$ through

$$\hat{R}^\dagger(\phi) \hat{\Delta}(x, p) \hat{R}(\phi) = \hat{\Delta}(X_\phi, Y_\phi), \quad (12)$$

where the rotated labels (X_ϕ, Y_ϕ) are the c-number quadratures obtained from (x, p)

$$X_\phi = x \cos \phi + p \sin \phi, \quad (13a)$$

$$Y_\phi = p \cos \phi - x \sin \phi. \quad (13b)$$

It therefore follows that for all ϕ ,

$$\langle x_-, p_+ | \hat{\Pi}_\phi(\xi) | x_+, p_- \rangle = e^{i\gamma} \delta(\xi - X_\phi), \quad (14)$$

and hence the amplitude vanishes unless $\xi = X_\phi$. We note that this result is the natural extension to the continuum of a known connection between the king's problem and discrete Weyl and Wigner distributions [12].

Finally, for a rigorous statement of probabilities in the continuous variable case, we need to describe the king's projective measurement as the limit of a sequence of POVM measurements. Let the POVM $\{\hat{E}(\xi|\phi, \epsilon) | \xi \in \mathbb{R}\}$ be any operator-valued δ -sequence,

$$\int_{-\infty}^{\infty} d\xi \hat{E}(\xi|\phi, \epsilon) = \mathbb{1}, \quad \lim_{\epsilon \rightarrow 0} \hat{E}(\xi|\phi, \epsilon) = \delta(\xi - \hat{X}_\phi), \quad (15)$$

admitting a $\mathcal{B}(\phi)$ -diagonal Kraus decomposition $\hat{E}(\xi|\phi, \epsilon) = \sum_i \hat{A}_i^\dagger(\xi|\phi, \epsilon) \hat{A}_i(\xi|\phi, \epsilon)$ with

$$\hat{A}_i(\xi|\phi, \epsilon) = \int_{-\infty}^{\infty} d\xi' a_i(\xi, \xi'|\phi, \epsilon) \hat{\Pi}_\phi(\xi'). \quad (16)$$

When conditioned on initial and final states, the outcome distribution of the POVM measurement is then given by

$$P(\xi|\phi, \epsilon) = \frac{\sum_i |\langle x_-, p_+ | \hat{A}_i(\xi|\phi, \epsilon) | x_+, p_- \rangle|^2}{\sum_i \int_{-\infty}^{\infty} d\xi |\langle x_-, p_+ | \hat{A}_i(\xi|\phi, \epsilon) | x_+, p_- \rangle|^2}. \quad (17)$$

Using Eq. (14), we find the amplitudes

$$|\langle x_-, p_+ | \hat{A}_i(\xi|\phi, \epsilon) | x_+, p_- \rangle|^2 = |a_i(\xi, X_\phi|\phi, \epsilon)|^2, \quad (18)$$

and hence the outcome distribution

$$P(\xi|\phi, \epsilon) = Z^{-1} \sum_i |a_i(\xi, X_\phi|\phi, \epsilon)|^2, \quad (19)$$

where Z is a normalization constant. Then, from the condition $\lim_{\epsilon \rightarrow 0} \hat{E}(\xi|\phi, \epsilon) = \delta(\xi - \hat{X}_\phi)$ and the diagonality of the Kraus operators, the condition $\lim_{\epsilon \rightarrow 0} \sum_i |a_i(\xi, X_\phi|\phi, \epsilon)|^2 = \delta(\xi - X_\phi)$ follows. Thus,

$$\lim_{\epsilon \rightarrow 0} P(\xi|\phi, \epsilon) = \delta(\xi - X_\phi), \quad (20)$$

which is the desired result. So indeed, if Alice makes initial and final projective measurements of the commuting pairs (\hat{x}_+, \hat{p}_-) and (\hat{x}_-, \hat{p}_+) respectively, she will be able to retrodict the outcome X_ϕ of the king's intermediate measurement of any \hat{X}_ϕ for all values of ϕ , in the limit of an infinitely sharp measurement.

We have thus seen how, in conformity with the standard formulation of the king's problem, a value X_ϕ is assigned to every possible \hat{X}_ϕ in a context of *exclusive disjunction*; in other words, the assignment makes reference to a single intermediate measurement that is performed by the king at the exclusion of the other possibilities. One may then enquire as to whether there exists a “dual” measurement context consistent with the assignment of a value X_ϕ to every \hat{X}_ϕ , but in *conjunction*. This situation was partially probed in Ref. [8], where the case of two incompatible dense observables was considered. The results, however, have a natural extension to any number of linear combinations of the two observables. This extension therefore defines the dual measurement context for the continuous variable king's problem—what we henceforth term the “conjunctive” version of the problem.

The conjunctive version of the mean king's problem has to do with what at first sight would appear to be a truly impossible (if not ill-defined) inference task according to the complementarity heuristics. In this case, the king measures, simultaneously and *sharply*, a whole set of quadrature operators $\hat{X}_{\phi_1}, \hat{X}_{\phi_2}, \dots, \hat{X}_{\phi_n}$, with an operational definition of such a sharp measurement procedure to be given shortly. Alice is again put to the task of guessing the outcomes of the measurement, and with the same initial and final measurements discussed earlier, we shall see that she is able to retrodict all the outcomes of the king's single joint measurement.

We define the simultaneous sharp measurement (SSM) of a set of non-commuting quadrature operators $\hat{\mathbf{X}} = (\hat{X}_{\phi_1}, \hat{X}_{\phi_2}, \dots, \hat{X}_{\phi_n})$ in terms of a POVM measurement with continuous index set $\boldsymbol{\xi} = (\xi_1, \xi_2, \dots, \xi_n)$ representing the measurement outcomes. The POVM is assumed to admit a Kraus decomposition $\hat{E}(\boldsymbol{\xi}|\epsilon) = \sum_i \hat{A}_i^\dagger(\boldsymbol{\xi}|\epsilon) \hat{A}_i(\boldsymbol{\xi}|\epsilon)$ with Kraus operators of the form

$$\hat{A}_i(\boldsymbol{\xi}|\epsilon) = : f_i(\boldsymbol{\xi} - \hat{\mathbf{X}}|\epsilon) :, \quad (21)$$

where $: \dots :$ stands for the generalized Weyl-symmetric ordering

$$: f_i(\boldsymbol{\xi} - \hat{\mathbf{X}}|\epsilon) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} d^n \boldsymbol{\xi}' \int_{\mathbb{R}^n} d^n \boldsymbol{\chi} e^{-i\boldsymbol{\chi} \cdot (\boldsymbol{\xi} - \hat{\mathbf{X}} - \boldsymbol{\xi}')} f_i(\boldsymbol{\xi}'|\epsilon), \quad (22)$$

and where the $f_i(\boldsymbol{\xi}|\epsilon)$ are functions defining an n -dimensional δ -sequence in the sense that

$$\lim_{\epsilon \rightarrow 0} \sum_i |f_i(\boldsymbol{\xi}|\epsilon)|^2 = \delta^{(n)}(\boldsymbol{\xi}). \quad (23)$$

This choice of POVM arises naturally from a measurement implemented by n canonical instruments described by some initial state $\hat{\rho}$, with pointer variables $\hat{\boldsymbol{\xi}}$, and simultaneously coupled to the system through an impulsive Hamiltonian

$$\hat{H} = -\delta(t) \hat{\boldsymbol{\chi}} \cdot \hat{\mathbf{X}}, \quad (24)$$

where the $\hat{\chi}_i$ are conjugate to the pointer variables $([\hat{\chi}_i, \hat{\xi}_j] = i\delta_{ij})$. With an arbitrary decomposition of $\hat{\rho} = \sum_i p_i |\phi_i\rangle \langle \phi_i|$, the Kraus operators are then given by

$$\hat{A}_i(\boldsymbol{\xi}|\epsilon) = \sqrt{p_i} \langle \boldsymbol{\xi} | e^{i\hat{\boldsymbol{\chi}} \cdot \hat{\mathbf{X}}} | \phi_i \rangle. \quad (25)$$

“Sharpness” in this context therefore means that the δ -sequence is defined from $\langle \boldsymbol{\xi} | \hat{\rho} | \boldsymbol{\xi} \rangle = \sum_i |f_i(\boldsymbol{\xi}|\epsilon)|^2$, the pre-measurement probability distribution of the pointer variables, as in the case of a projective measurement.

Now, in accordance with the uncertainty principle, the mutual back-reaction between the different instruments ensures that the probability distribution of the final readings from any preselected ensemble will necessarily be unsharp [13]. To see this, note that the SSM can be described in the Heisenberg picture as the action of the operator $\hat{U} = e^{i\hat{\boldsymbol{\chi}} \cdot \hat{\mathbf{X}}}$ on the instrument pointer variables $\hat{\boldsymbol{\xi}}$. The pointer operators representing the final outcome can then be written as

$$\hat{\boldsymbol{\xi}} = \hat{U}^\dagger \hat{\boldsymbol{\xi}} \hat{U} |_{\text{in}} = \hat{\boldsymbol{\xi}}_{\text{in}} + \hat{\mathbf{X}}_{\text{in}} + \frac{1}{2} \mathbf{C} \hat{\boldsymbol{\chi}}_{\text{in}} \quad (26)$$

where “in” denotes the pre-measurement Heisenberg variables and where \mathbf{C} is the antisymmetric matrix with elements

$$C_{ij} = i[\hat{X}_{\phi_i}, \hat{X}_{\phi_j}] = \sin(\phi_i - \phi_j), \quad (27)$$

arising from the commutator of the quadrature operators. The back-reaction of the instruments is reflected in a dependence of the outcomes $\hat{\boldsymbol{\xi}}$ on the conjugate variables $\hat{\boldsymbol{\chi}}_{\text{in}}$, which are uncertain to the same extent that the initial pointer variables $\hat{\boldsymbol{\xi}}_{\text{in}}$ are certain. In particular, in the “sharp” limit $\epsilon \rightarrow 0$, the uncertainties in the readings from any preselected ensemble are dominated by the divergent uncertainties of the conjugate $\hat{\boldsymbol{\chi}}_{\text{in}}$ variables. The prospect of any precise inference of the joint outcomes would therefore seem to be doomed based on the mutual disturbance of the simultaneous measurements.

Let us, however, compute the posterior probability distribution of outcomes of the SSM, when conditioned on

the initial and final states previously discussed. The probability distribution is then given by

$$P(\xi|\epsilon) = Z^{-1} \sum_i \left| \langle x_-, p_+ | \hat{A}_i(\xi|\epsilon) | x_+, p_- \rangle \right|^2. \quad (28)$$

Given the definition of $\hat{A}_i(\xi|\epsilon)$ in terms of the Weyl-symmetrization prescription, (22), the amplitude $\langle x_-, p_+ | \hat{A}_i(\xi|\epsilon) | x_+, p_- \rangle$ is the convolution of $f_i(\xi|\epsilon)$ and the Fourier transform of

$$\langle x_-, p_+ | e^{i\mathbf{x} \cdot \hat{\mathbf{X}}} | x_+, p_- \rangle. \quad (29)$$

Concentrating on this matrix element, we write the exponent $\mathbf{x} \cdot \hat{\mathbf{X}}$ as

$$\mathbf{x} \cdot \hat{\mathbf{X}} = a\hat{x} + b\hat{p} \quad (30)$$

for some \mathbf{x} -dependent a and b . Next we note that the linear combination $a\hat{x} + b\hat{p}$ can also be written in terms of the collective canonical variables $(\hat{x}_\pm, \hat{p}_\pm)$ as

$$a\hat{x} + b\hat{p} = \hat{L}(a, b) + \hat{R}(a, b), \quad (31)$$

where $\hat{L}(a, b)$ and $\hat{R}(a, b)$ are *commuting* operators defined as

$$\hat{L}(a, b) = (a\hat{x}_- + b\hat{p}_+)/2, \quad (32)$$

$$\hat{R}(a, b) = a\hat{x}_+ + b\hat{p}_-. \quad (33)$$

Consequently, the operator exponential can be factored as

$$e^{i(a\hat{x} + b\hat{p})} = e^{i\hat{L}(a, b)} e^{i\hat{R}(a, b)}. \quad (34)$$

Since the eigenvectors $|x_-, p_+\rangle$ and $|x_+, p_-\rangle$ are respectively eigenvectors of $\hat{L}(a, b)$ and $\hat{R}(a, b)$, we then verify that

$$\langle x_-, p_+ | e^{i(a\hat{x} + b\hat{p})} | x_+, p_- \rangle = e^{i\gamma} e^{i(ax + bp)}, \quad (35)$$

where x and p are as defined in Eq. (8), or equivalently,

$$\langle x_-, p_+ | e^{i\mathbf{x} \cdot \hat{\mathbf{X}}} | x_+, p_- \rangle = e^{i\gamma} e^{i\mathbf{x} \cdot \mathbf{X}}, \quad (36)$$

where $\mathbf{X} = (X_{\phi_1}, X_{\phi_1}, \dots, X_{\phi_n})$ with the X_ϕ as defined in Eq. (13a). The Fourier transform of the matrix element is therefore a δ -function at \mathbf{X} , so that upon convolution, we find that

$$\langle x_-, p_+ | \hat{A}_i(\xi|\epsilon) | x_+, p_- \rangle = e^{i\gamma} f_i(\xi - \mathbf{X}|\epsilon). \quad (37)$$

The joint outcome probability distribution for the measurement is then

$$P(\xi|\epsilon) = \sum_i |f_i(\xi - \mathbf{X}|\epsilon)|^2, \quad (38)$$

and with the δ -sequence condition (23), we finally obtain

$$\lim_{\epsilon \rightarrow 0} P(\xi|\epsilon) = \delta^{(n)}(\xi - \mathbf{X}). \quad (39)$$

Alice is therefore able to ascertain the king's joint measurement outcomes in the sharp limit of the conjunctive version of the challenge, as originally advertised.

The exact coincidence between the conditional outcomes of the conjunctive version of the king's problem and the outcomes in the standard version would seem to suggest that Alice's assignments are, in some sense, non-contextual (independent of measurement conditions). However, as in other situations where conditioning from postselection allows precise assignments to incompatible measurement outcomes [14, 15], it is also possible to find for the same pre and postselections, potential intermediate measurements exhibiting contextuality in the present case. An interesting example highlighting the role of entanglement has to do with the intermediate SSM measurement of two sets of quadrature-operator arrays, $\hat{\mathbf{X}}$ and $\hat{\mathbf{X}}'$, the second set involving operators of Alice's ancillary particle. By the symmetry of the problem, sharp conditional probabilities can also be assigned to the SSM measurement outcomes of $\hat{\mathbf{X}}'$, but only in the absence of measurements on the original particle. On the other hand (and as discussed briefly in [8]), a non-sharp conditional probability distribution generally follows for the full outcome set in the SSM of $\hat{\mathbf{X}}$ and $\hat{\mathbf{X}}'$, showing that the assignment of sharp values to either $\hat{\mathbf{X}}$ or $\hat{\mathbf{X}}'$ is, in fact, contextual. Nonetheless, our results show that entanglement allows for a surprising degree of flexibility in the composition of the observable sets that reveal the contextual aspects of retrodiction from postselection.

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